MATH3310 Tutorial 10 Questions

1. We consider a modified power iteration. For any $\mathbf{x} = (x_1, x_2, ..., x_n)^T \in \mathbb{C}^n$, we define: $g(\mathbf{x}) = x_l$, where l is the smallest index such that $|x_l| = ||\mathbf{x}||_\infty$. Let $A \in M_{n\times n}(\mathbb{C})$ whose eigenvalues are given by $\lambda_1, \lambda_2, ...\lambda_n$ such that

$$|\lambda_1| > |\lambda_2| > ... > |\lambda_n| > 0.$$

Suppose $\mathbf{u}_i$ is the eigenvector of A corresponding to the eigenvalue λ_i for $i = 1, 2, ..., n$. Let $\mathbf{x}_0 = \sum_{i=1}^n \mathbf{u}_i$. The modified power iteration can be written as:

$$\mathbf{x}_{k+1} = \frac{A\mathbf{x}_k}{g(A\mathbf{x}_k)}.$$

   (a) With the above modified power iteration, prove that $\lim_{k\to\infty} g(A\mathbf{x}_k) = \lambda_1$. Please explain your answer with details.

   (b) The modified inverse power iteration can be formulated as

$$\mathbf{x}_{k+1} = \frac{A^{-1}\mathbf{x}_k}{g(A^{-1}\mathbf{x}_k)}.$$

   With the same $\mathbf{x}_0$ defined as above, prove that $\lim_{k\to\infty} g(A\mathbf{x}_k) = \lambda_n$.

2. Let A be a non-singular $n \times n$ real matrix. We apply the QR method on A to obtain a sequence of matrices $\{A^{(j)}\}_{j=0}^\infty$, which satisfies:

$$A^{(0)} = A;$$
$$A^{(j+1)} = R^{(j)}Q^{(j)} \text{ for } j = 0, 1, 2, ...,$$

where $A^{(j)} = Q^{(j)}R^{(j)}$ is the QR factorization of $A^{(j)}$. Let k be an integer greater than 2020. Given that the QR factorizations of A^{k-1} and $A^{(k-1)}$ are given by

$$A^{k-1} = Q_1R_1 \text{ and } A^{(k-1)} = Q_2R_2.$$

In this question, all QR factorization is obtained in such a way that the diagonal entries of the upper triangular matrix are postive.

   (a) Express A in terms of Q_1, Q_2, R_1 and R_2 only. Please explain your answer with details.

   (b) Starting from $\mathbf{x}_0$, we apply the Power's method on A as follows:

$$\mathbf{x}_{j+1} = \frac{A\mathbf{x}_j}{||A\mathbf{x}_j||_\infty} \text{ for } j = 0, 1, 2, ...$$

   Write $\mathbf{x}_k$ in terms of $\mathbf{x}_0$, Q_1, Q_2, R_1 and R_2 only (without A and k). Please explain your answer with details.

3. Consider the gradient descent algorithm to solve the linear system $A\mathbf{x} = \mathbf{b}$, where A is a $n \times n$ symmetric positive definite real matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$. Define the convex quadratic objective function $f(\mathbf{x})$ as follows:

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}.$$

The gradient descent algorithm can be written as:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \lambda_k \mathbf{d}_k, \text{ where } \mathbf{d}_k = \nabla f(\mathbf{x}_k) \text{ and } \lambda_k = \frac{\mathbf{d}_k^T \mathbf{d}_k}{\mathbf{d}_k^T A \mathbf{d}_k}.$$

Define: $||\mathbf{x}||_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$ and $||\mathbf{x}||_A = \sqrt{\mathbf{x}^T A\mathbf{x}}$.

Denote the eigenvalues of A by $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_n$. For any nonzero $\mathbf{x} \in \mathbb{R}^n$, we are given the following inequality:

$$\frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \cdot \frac{\mathbf{x}^T A^{-1}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \frac{(\mu_1 + \mu_n)^2}{4\mu_1\mu_n}$$

Introduce the error function as $E(\mathbf{x}) = ||\mathbf{x} - \mathbf{x}^*||_A^2$, where $\mathbf{x}^*$ is the solution of $A\mathbf{x} = \mathbf{b}$. Define $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}^*$.

(a) Prove that $\mathbf{e}_{k+1} = \mathbf{e}_k - \lambda_k \mathbf{d}_k$ and $\mathbf{e}_k = A^{-1}\mathbf{d}_k$.

(b) Prove that:

$$||\mathbf{e}_{k+1}||_A \leq \left(\frac{c(A) - 1}{c(A) + 1}\right) ||\mathbf{e}_k||_A.$$

where $c(A) = \frac{\mu_n}{\mu_1}$ is the condition number of A.